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EXACT MULTIPLICITY AND STABILITY OF SOLUTIONS OF A 1-DIMENSIONAL, p-LAPLACIAN PROBLEM WITH POSITIVE CONVEX NONLINEARITY

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ABSTRACT. We consider the p-Laplacian problem

$$-\phi_p(u')' = \lambda f(u), \quad \text{on } (-1,1), \tag{1}$$

$$u(\pm 1) = 0, \tag{2}$$

where $p > 1$ ($p \neq 2$), $\phi_p(z) := |z|^{p-1} \operatorname{sgn} z$, $z \in \mathbb{R}$, $\lambda \geqslant 0$, and $f : \mathbb{R} \to (0, \infty)$ is C^2 and there exists $C > 0$, $q > p$ such that

$$f(\xi) \geqslant C(1 + \xi^{q-1}),\ f'(\xi) > 0,\ f''(\xi) > 0,\ \xi > 0.$$

We show that the set of solutions (λ, u) of (1)-(2) consists of a C^2 curve in $(0,\infty) \times C^1[-1,1]$ with end point $(0,0) \in \mathbb{R} \times C^1[-1,1]$ and which tends to $\{0\} \times \infty$, and has a single turning point. Thus, there exists $\lambda^* > 0$ such that (1)-(2) has exactly two solutions, $u^-(\lambda) < u^+(\lambda)$, when $0 < \lambda < \lambda^*$, and no solutions when $\lambda > \lambda^*$.

When $0 < \lambda < \lambda^*$ the solutions $u^\pm(\lambda)$ are equilibria of a related time-dependent, parabolic problem, and in this context their stability is of interest. We show that the 'lower' solution $u^-(\lambda)$ is stable and the 'upper' solution $u^+(\lambda)$ is unstable, and solutions of the parabolic problem with initial values below $u^+(\lambda)$ converge to $u^-(\lambda)$, while those with initial values above $u^+(\lambda)$ are unbounded.

1. Introduction

We consider the p-Laplacian boundary-value problem

$$-\phi_p(u')' = \lambda f(u), \quad \text{on } (-1,1), \tag{1.1}$$

$$u(\pm 1) = 0, \tag{1.2}$$

where $p > 1$ ($p \neq 2$), $\phi_p(z) := |z|^{p-1} \operatorname{sgn} z$, $z \in \mathbb{R}$, $\lambda \geqslant 0$, and the function $f : \mathbb{R} \to (0, \infty)$ is C^2, strictly positive and there exists $C > 0$, $q > p$ such that, for all $\xi > 0$:

$$f(\xi) \geqslant C(1 + \xi^{q-1}), \tag{1.3}$$

$$f'(\xi) > 0,\ \ f''(\xi) > 0. \tag{1.4}$$

It will be shown that the set of solutions (λ, u) of (1.1)-(1.2) consists of a C^2 curve $\mathcal{S}$ in $(0,\infty) \times C^1[-1,1]$ with an end point at $(0,0) \in \mathbb{R} \times C^1[-1,1]$ and which tends to $\{0\} \times \infty$, and having a single 'turning point', or 'fold bifurcation' (these statements will be made precise below). Thus, there exists $\lambda^* > 0$ such that (1.1)-(1.2) has exactly two solutions $u^\pm(\lambda)$, when $0 < \lambda < \lambda^*$, and no solutions when $\lambda > \lambda^*$. In addition, $u^-(\lambda) < u^+(\lambda)$ on $(-1,1)$, for all $0 < \lambda < \lambda^*$.

The solutions $u^\pm(\lambda)$, $0 < \lambda < \lambda^*$, of (1.1)-(1.2) can also be regarded as equilibrium solutions of a related time-dependent, parabolic initial value problem. In this setting the stability of these equilibria is of interest, and this will also be determined. It will be shown that the 'lower' solution $u^-(\lambda)$ is stable and the 'upper' solution $u^+(\lambda)$ is unstable. In fact, it will be shown that solutions of the parabolic problem with initial values below $u^+(\lambda)$ converge to $u^-(\lambda)$, while those with initial values above $u^+(\lambda)$ are unbounded.

Similar results on the structure of the solution set $\mathcal{S}$ (but not the stability properties of the solutions) are obtained in [1, Theorem 3.9], for a similar ODE problem, where the corresponding ordinary differential operator is obtained from the radial p-Laplacian operator in $\mathbb{R}^N$, with $N \geqslant 4$, under the assumption that $1 < p < 2$, together with an additional, restrictive, growth condition on f. The paper [1] contains a detailed survey of previous results and literature on the structure of $\mathcal{S}$ for similar problems (including similar problems in $\mathbb{R}^N$, $N \geqslant 1$), and in particular for the semilinear case $p = 2$. In view of this we will say no more about the background to this problem here, and simply refer to [1] for further details. However, we will make some further comparisons of our results with those of [1] below; of course, our results do not overlap with those of [1], since our operator corresponds to $N = 1$, whereas $N \geqslant 4$ is assumed in [1].

2. Preliminaries

2.1. **Some notation.** For any integer $j \geqslant 0$, $C^j[-1,1]$ will denote the standard Banach space of real valued, j-times continuously differentiable functions ω defined on $[-1,1]$, with the norm $|\omega|_j = \sum_{i=0}^{j} |\omega^{(i)}|_0$, where $|\cdot|_0$ denotes the usual sup-norm on $C^0[-1,1]$ (throughout, all function spaces will be real). For any $r \geqslant 1$, $L^r(-1,1)$ will denote the standard Banach space of real valued functions on $[-1,1]$ whose rth power is integrable, with norm $\|\cdot\|_r$. Also, $W^{1,r}(-1,1)$, with norm $\|\cdot\|_{1,r}$, will denote the usual Sobolev space of absolutely continuous functions ω on $[-1,1]$, with derivative $\omega' \in L^r(-1,1)$. We also let $C_0^j[-1,1]$, $W_0^{1,r}(-1,1)$ denote the set of functions ω in $C^j[-1,1]$, $W^{1,r}(-1,1)$, respectively, satisfying the boundary conditions (1.2).

If $F : X \to Z$ is a function between Banach spaces X and Z, then $DF(x) : X \to Z$ will denote the Fréchet derivative of F at x; partial Fréchet derivatives will be indicated by subscripts, for example, $D_xG(x,y)$, $D_yG(x,y)$ will denote the partial derivatives of a function G depending on x and y. The second Fréchet derivative of F at x will be denoted by $D^2F(x) : X \times X \to Z$.

For any continuous function $g : \mathbb{R} \to \mathbb{R}$, and any $\omega \in C^0[-1,1]$, we define $g(\omega) \in C^0[-1,1]$ by $g(\omega)(x) = g(\omega(x))$, $x \in [-1,1]$ (that is, g will denote both a function and its corresponding Nemytskii operator).

In connection with the inverse p-Laplacian below, it will also be convenient to define the number

$$p^* := \frac{1}{p-1} > 0.$$

2.2. **Basic solution properties.** We begin with some basic properties of solutions of the problem (1.1)–(1.2). The first result follows immediately from parts (α) and (β) of [8, Theorem 4] – we state it here for reference below.

Lemma 2.1. *If the initial value problem consisting of the differential equation* (1.1)*, with* $\lambda \geqslant 0$*, together with initial conditions* $u(x_0) = \alpha$*,* $u'(x_0) = \beta$*,* $x_0 \in [-1,1]$*,* $\alpha, \beta \in \mathbb{R}$*, has a solution on some interval containing* x_0 *then this solution is unique.*

We now define the *p-Laplacian* operator $\Delta_p : D(\Delta_p) \subset C_0^1[-1,1] \to C^0[-1,1]$ by

$$D(\Delta_p) := \{u \in C_0^1[-1,1] : \phi_p(u') \in C^1[-1,1]\},$$
$$\Delta_p(u) := \phi_p(u')', \quad u \in D(\Delta_p).$$

This operator is $(p-1)$-*homogeneous*, that is, $\Delta_p(tu) = \phi_p(t)\Delta_p(u)$, for any $t \in \mathbb{R}$ and $u \in D(\Delta_p)$. With this notation the problem (1.1)–(1.2) can be rewritten as

$$-\Delta_p(u) = \lambda f(u), \quad (\lambda, u) \in [0,\infty) \times D(\Delta_p). \tag{2.1}$$

Clearly, $(\lambda, u) = (0,0)$ is a solution of (2.1) and, by the strict positivity of f and Lemma 4.1, the only solution of (2.1) with $\lambda = 0$ or $u = 0$ is the trivial solution $(0,0)$. Let

$$\mathcal{S} := \{(\lambda, u) \in (0,\infty) \times D(\Delta_p) \text{ satisfying (2.1)}\}.$$

Lemma 2.2. *Suppose that* $(\lambda, u) \in \mathcal{S}$*. Then:*

(*a*) $u > 0$ *on* $(-1,1)$ *and* $\pm u'(\pm 1) < 0$;
(*b*) u *is symmetric about* $x = 0$ (*that is,* u *is even*)*, and* $u'(0) = 0$;
(*c*) $u \in C^2(0,1]$*, and* $u' < 0$*,* $u'' < 0$ *on* $(0,1]$.

Proof. It follows from the differential equation (1.1) and the strict positivity of f that u cannot have a local minimum in the interval $(-1,1)$, so $u > 0$ on $(-1,1)$, and hence, by Lemma 2.1, $\pm u'(\pm 1) < 0$, which proves part (a). It now follows that u must have a local maximum at some $x_0 \in (-1,1)$, and by (1.1), $u' > 0$ on $(-1,x_0)$, and $u' < 0$ on $(x_0,1)$. Furthermore, since f is independent of x, it follows from Lemma 2.1 that u must be symmetric about x_0, and since $u > 0$

on $(-1,1)$ and $u(\pm 1)=0$, we must have $x_0=0$, which proves part (b) and the first inequality in part (c). It now follows from that inequality that we can write the basic equation (4.1) as

$$\big(|u_0'|^{p-2}u_0'\big)' = (p-1)|u_0'|^{p-2}u_0'' = -\lambda_0 f(u_0), \quad \text{on } (0,1], \tag{2.2}$$

from which we see that $u \in C^2(0,1]$ and the second inequality in part (c) holds on $(0,1]$. □

3. The structure of the solution set $\mathcal{S}$

We can now state our main results regarding the structure of the solution set $\mathcal{S}$ – viz., that $\mathcal{S}$ is a smooth curve, with certain properties. In these results a 'parametrisation' $s:\mathcal{I}\to(\lambda(s),u(s))$, on an open interval $\mathcal{I}\subset\mathbb{R}$, will mean a smooth mapping with the tangent vector $(\lambda_s(s),u_s(s))\neq(0,0)$, $s\in\mathcal{I}$ (the smoothness and range spaces will be specified in each case). We use the notation λ_s and u_s to denote the derivatives with respect to s, to avoid confusion with the x derivative of the function $u(s)\in C_0^1[-1,1]$, which will be denoted $u(s)'$. Also, the value of $u(s)$ at $x=0$ will be denoted by $u(s)|_0$.

Theorem 3.1. *(A) The set $\mathcal{S}$ consists of a single (connected) C^2 curve in $(0,\infty)\times C_0^1[-1,1]$, having a parametrisation of the form $s:(0,\infty)\to(\lambda(s),u(s))$ with the properties:*

(a) $\lim_{s\to 0^+}(\lambda(s),u(s))=(0,0)$, $\quad \lim_{s\to\infty}\lambda(s)=0$, $\quad \lim_{s\to\infty}|u(s)|_0=\infty$;

(b) there exists a unique $s^*\in(0,\infty)$ such that

$$\pm(s-s^*)>0 \implies \pm\lambda'(s)<0, \quad \lambda'(s^*)=0, \quad \lambda''(s^*)<0;$$

(c) the function $s\to|u(s)|_0=u(s)|_0$ is strictly increasing.

(B) In a neighbourhood of $(\lambda,u)=(0,0)$ in $\mathbb{R}\times C_0^1[-1,1]$, the set of solutions of (2.1) has the form

$$\{(\lambda,u): u=U(\lambda^{p^*}),\ \lambda^{p^*}\in(-\epsilon,\epsilon)\},$$

where $\epsilon>0$ and the function $U:(-\epsilon,\epsilon)\to C_0^1[-1,1]$ is C^2.

Theorem 3.1 describes the shape of the curve $\mathcal{S}$ in $(0,\infty)\times C_0^1[-1,1]$. An alternative description is given in the following corollary.

Corollary 3.2. *The set $\mathcal{S}\setminus\{(\lambda(s^*),u(s^*))\}$ consists of a pair of C^2 curves in $(0,\infty)\times C_0^1[-1,1]$ of the form*

$$\mathcal{S}^{\pm}=\{(\lambda,u^{\pm}(\lambda)):\lambda\in(0,\lambda^*)\},$$

where the mappings $u^{\pm}:(0,\lambda^*)\to C_0^1[-1,1]$ are C^2, and

$$\lim_{\lambda\to 0^+}u^-(\lambda)=0, \quad \lim_{\lambda\to 0^+}u^+(\lambda)|_0=\infty, \quad \lim_{\lambda\to\lambda^{*-}}u^{\pm}(\lambda)=u(s^*).$$

Furthermore,

$$u^-(\lambda)<u^+(\lambda), \quad \text{on } (-1,1), \quad \text{for all } \lambda\in(0,\lambda^*). \tag{3.1}$$

Hence:

- *if $0<\lambda<\lambda^*$ then (2.1) has exactly two solutions*;
- *if $\lambda=\lambda^*$ then (2.1) has exactly one solution*;
- *if $\lambda^*<\lambda$ then (2.1) has no solution.*

Remark 3.3. The results of Theorem 3.1 and Corollary 3.2 are proved in [1, Theorem 3.9], under the assumptions: $1<p<2$, $N\geqslant 4$, and an additional growth condition on f.

4. The proof of Theorem 3.1 and Corollary 3.2

It follows from Lemma 2.2 that equation (2.1) is equivalent to the problem

$$-\phi_p(u')' = \lambda f(u), \quad \text{on } (0,1), \tag{4.1}$$

$$u'(0) = 0, \quad u(1) = 0, \tag{4.2}$$

and it will be convenient to consider the problem in this formulation. To facilitate this, we define analogues of some of the above constructions on the interval $[0,1]$. We let

$$D(\widehat{\Delta}_p) := \{u \in C^1[0,1] : u \text{ satisfies (4.2) and } \phi_p(u') \in C^1[0,1]\},$$

$$\widehat{\Delta}_p(u) := \phi_p(u')', \quad u \in D(\widehat{\Delta}_p),$$

and we rewrite (4.1)-(4.2) in the form

$$-\widehat{\Delta}_p(u) = \lambda f(u), \quad (\lambda, u) \in [0,\infty) \times D(\widehat{\Delta}_p), \tag{4.3}$$

and define

$$\widehat{\mathcal{S}} := \{(\lambda, u) \in (0,\infty) \times D(\widehat{\Delta}_p) \text{ satisfying (4.3)}\}.$$

In view of (4.3) and the strict positivity of f, we begin by considering the boundary value problem

$$-\widehat{\Delta}_p(u) = h, \quad h \in P^0, \tag{4.4}$$

where

$$P^0 := \{h \in C^0[0,1] : h > 0 \text{ on } [0,1]\} \subset C^0[0,1],$$

and we construct a 'solution operator' for the problem (4.4). To do this we first define bounded, linear, integral operators I, $J : C^0[0,1] \to C^1[0,1]$ by

$$I(h)(x) := \int_x^1 h(t)\,dt, \quad J(h)(x) := \int_0^x h(t)\,dt, \quad x \in [0,1], \quad h \in C^0[0,1]. \tag{4.5}$$

Clearly, if $h \in P^0$ then $J(h) \geqslant 0$ on $[0,1]$. It is now easy to construct the (unique) solution of (4.4). For reference, we describe this in the following lemma.

Lemma 4.1. *For any $h \in P^0$ the problem (4.4) has a unique solution $S_p(h) \in C^1[0,1]$ given by*

$$S_p(h) := I\big\{J(h)^{p^*}\big\}. \tag{4.6}$$

The operator $S_p : C^0[0,1] \to C^1[0,1]$ is continuous and p^*-homogeneous.

4.1. **The differentiability of S_p.** We now show that the operator S_p is C^2 on P^0 (which is clearly an open subset of $C^0[0,1]$). We first observe that S_p was constructed by solving the problem (4.4), and in this context it is naturally expressed in terms of the operator J, as in (4.6). However, to derive the differentiability of S_p it will be convenient to rewrite the operator J slightly. For any $h \in P^0$ and $x \in [0,1]$,

$$J(h)(x) = x\widetilde{J}(h)(x), \quad \text{where } \widetilde{J}(h)(x) := \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x h(t)\,dt, & x \in (0,1], \\ h(0), & x = 0. \end{cases}$$

The linear operator $\widetilde{J} : C^0[0,1] \to C^0[0,1]$ is bounded, and $\widetilde{J}(P^0) \subset P^0$. Using $\widetilde{J}$ we can now reformulate (4.6) as

$$S_p(h) = I\big\{J(h)^{p^*}\big\} = I\big\{x^{p^*}\widetilde{J}(h)^{p^*}\big\}, \quad h \in P^0 \tag{4.7}$$

(for simplicity, in (4.7) we use the notation x^{p^*} to denote the function $x \to x^{p^*}$ on $[0,1]$).

Theorem 4.2. (*A*) *For any $p > 1$ the mapping $S_p : P^0 \to C^1[0,1]$ is C^2. For any $h \in P^0$, the derivatives of S_p at h are given by*

$$DS_p(h)\bar{h} = p^* I\big\{x^{p^*}\widetilde{J}(h)^{p^*-1}\widetilde{J}(\bar{h})\big\}, \qquad \bar{h} \in C^0[0,1], \tag{4.8}$$

$$D^2S_p(h)(\bar{h}_1, \bar{h}_2) = p^*(p^*-1) I\big\{x^{p^*}\widetilde{J}(h)^{p^*-2}\widetilde{J}(\bar{h}_1)\widetilde{J}(\bar{h}_2)\big\}, \quad \bar{h}_1, \bar{h}_2 \in C^0[0,1]. \tag{4.9}$$

(*B*) *Suppose that* $h \in P^0$, $u = S_p(h)$, $\bar{h} \in C^0[0,1]$ *and* $w = DS_p(h)\bar{h} \in C^1[0,1]$. *Then* w', $|u'|^{p-2}w' \in C^1[0,1]$ (*see Remark 4.3 (b)*) *and*

$$\begin{gathered} -(|u'|^{p-2}w')' = p^*\bar{h}, \\ w'(0) = 0, \quad (|u'|^{p-2}w')|_0 = 0, \quad w(1) = 0, \end{gathered} \tag{4.10}$$

where $(|u'|^{p-2}w')|_0$ *denotes the value of the function* $|u'|^{p-2}w'$ *at* $x = 0$.

Remark 4.3. (a) In (4.8) and (4.9), the function $\widetilde{J}(h) \in P^0$, so the functions $\widetilde{J}(h)^{p^*-1}$, $\widetilde{J}(h)^{p^*-2} \in P^0$, even when $p^* < 1$.

(b) In part (B) of Theorem 4.2, if $1 < p < 2$ then the function $|u'|^{p-2}$ is singular at $x = 0$ (but nowhere else, by (4.4) and the assumption that $h \in P^0$), so the implication in this result is that the function $|u'|^{p-2}w'$ coincides with an element of $C^1[0,1]$ on $(0,1]$, so it extends smoothly to an element of $C^1[0,1]$, with a well-defined value at $x = 0$.

Proof. (A) We first evaluate the Gâteaux derivative of S_p at $h \in P^0$ (see [11, Definition 4.5]; we will denote this derivative by $D_G S_p(h)$). Suppose that $\bar{h} \in C^0[0,1]$, and $\delta \in \mathbb{R}$ is sufficiently small that $|\delta\bar{h}| < h$ on $[0,1]$. Then $|\widetilde{J}(\delta\bar{h})| < \widetilde{J}(h)$, so by (4.7) and the mean value theorem,

$$\begin{aligned} Q_{1,\delta} := \frac{1}{\delta}\big(S_p(h+\delta\bar{h}) - S_p(h)\big) &= \frac{1}{\delta} I\big\{x^{p^*}\big((\widetilde{J}(h) + \widetilde{J}(\delta\bar{h}))^{p^*} - \widetilde{J}(h)^{p^*}\big)\big\} \\ &= p^* I\big\{x^{p^*}(\widetilde{J}(h) + \delta\theta\widetilde{J}(\bar{h}))^{p^*-1}\widetilde{J}(\bar{h})\big\}, \end{aligned} \tag{4.11}$$

where $\theta(x) \in (0,1)$, $x \in [0,1]$. Now, by the form of the operator I,

$$\lim_{\delta\to 0} Q_{1,\delta} = p^* I\big\{x^{p^*}\widetilde{J}(h)^{p^*-1}\widetilde{J}(\bar{h})\big\}, \quad \lim_{\delta\to 0} Q'_{1,\delta} = -p^* x^{p^*}\widetilde{J}(h)^{p^*-1}\widetilde{J}(\bar{h}), \tag{4.12}$$

where the convergence in the limits in (4.12) is in $C^0[0,1]$, and so the derivative quotient $Q_{1,\delta}$ in (4.11) converges in $C^1[0,1]$. Hence, the Gâteaux derivative $D_G S_p(h)\bar{h}$ exists for all $h \in P^0$, $\bar{h} \in C^0[0,1]$, and, by (4.12), is given by (4.8). Furthermore, it is clear that the bounded, linear operator $D_G S_p(h) : C^0[0,1] \to C^1[0,1]$ depends continuously on $h \in P^0$ so, by [11, Proposition 4.8], S_p is continuously Fréchet differentiable on P^0, and the Fréchet derivative is given by (4.8).

Next, we evaluate the second Gâteaux derivative of S_p at $h \in P^0$ (we will denote this derivative by $D_G^2 S_p(h)$). Suppose that $\bar{h}_1, \bar{h}_2 \in C^0[0,1]$ and $\delta \in \mathbb{R}$ is sufficiently small that $|\delta\bar{h}_2| < h$. Then, by (4.8), and the mean value theorem,

$$\begin{aligned} Q_{2,\delta} := \frac{1}{\delta}\big(DS_p(h+\delta\bar{h}_2)\bar{h}_1 - DS_p(h)\bar{h}_1\big) &= \frac{p^*}{\delta} I\big\{x^{p^*}\big((\widetilde{J}(h) + \widetilde{J}(\delta\bar{h}_2))^{p^*-1} - \widetilde{J}(h)^{p^*-1}\big)\widetilde{J}(\bar{h}_1)\big\} \\ &= p^*(p^*-1) I\big\{x^{p^*}(\widetilde{J}(h) + \delta\theta\widetilde{J}(\bar{h}_2))^{p^*-2}\widetilde{J}(\bar{h}_1)\widetilde{J}(\bar{h}_2)\big\}. \end{aligned}$$

It is clear that the derivative quotient $Q_{2,\delta}$ also converges in $C^1[0,1]$, and hence the second Gâteaux derivative $D_G^2 S_p(h)(\bar{h}_1, \bar{h}_2)$ exists and is given by (4.9). Furthermore, the bounded, bilinear operator $D_G^2 S_p(h) : C^0[0,1]^2 \to C^1[0,1]$ depends continuously on $h \in P^0$, so S_p is twice continuously Fréchet differentiable on P^0 (again by [11, Proposition 4.8]).

(B) Differentiating the function $u = S_p(h)$ with respect to x, on the interval $(0,1]$, using the formula (4.6), yields

$$u' = -J(h)^{p^*} \implies |u'|^{2-p} = J(h)^{p^*(2-p)} = x^{p^*-1}\widetilde{J}(h)^{p^*-1} \in L^1(0,1) \tag{4.13}$$

(since $p^* > 0$), and differentiating $w = DS_p(h)\bar{h}$ with respect to x, on $(0,1]$, using the formula (4.8), and combining this with (4.13) yields

$$\begin{aligned} w' &= -p^* x^{p^*}\widetilde{J}(h)^{p^*-1}\widetilde{J}(\bar{h}) = -p^* J(h)^{p^*-1} J(\bar{h}) \\ &\implies |u'|^{p-2}w' = -p^* J(\bar{h}) \in C^1[0,1], \end{aligned}$$

and part (B) of the theorem now follows from these results and the form of the operator $DS_p(h)$ in (4.8), which completes the proof of Theorem 4.2. □

Remark 4.4. (a) The C^1 differentiability of the inverse of the p-Laplacian, on a suitable domain, is proved in [3, Theorem 3.4], for the case of periodic boundary conditions; the proof for other boundary conditions is similar. This is extended to the radial p-Laplacian in [5, 6, 7]. These theorems deal separately with the cases $1 < p < 2$ and $p > 2$. When $p > 2$ they require additional conditions at the zeros of u' (where $u = S_p(h)$) and they only obtain first order differentiability into the space $W^{1,1}(-1, 1)$ (rather than $C^1[0, 1]$). Thus, Theorem 4.2 is considerably stronger than these results, but this is due to the symmetry in the problem here, which allows for the factorisation of $J = s\widetilde{J}$, and the restriction of the domain of S_p to P^0.

(b) A similar result to Theorem 4.2-(A) on C^2 differentiability is proved in [1, Lemma 3.3] for an operator having a similar form to S_p above, but obtained from the radial p-Laplacian operator in the unit ball in $\mathbb{R}^N$, with $N \geqslant 1$, and with $1 < p < 2$. Although the case $1 < p < 2$ is considered in [1, Lemma 3.3], additional conditions at the zeros of u' are imposed, and second order differentiability into $W^{1,1}(0, 1)$ is claimed, although the proof only appears to show second order differentiability into the space $C^0[0, 1]$. However, the above (simpler) proof extends readily to the operator S_p considered in [1], to prove the result of [1, Lemma 3.3] for all $p > 1$, without the additional conditions imposed on u in [1, Lemma 3.3].

4.2. **Proof of Theorem 3.1.** Clearly, it suffices to show that the set $\widehat{\mathcal{S}}$ has all the properties described in the theorem. The proof of this follows a standard strategy, using the above constructions – we will prove some of the technical details specific to the problem considered here, and then simply sketch the rest of the proof.

A point $(\lambda_0, u_0) \in \widehat{\mathcal{S}}$ iff

$$F(\lambda_0, u_0) := u_0 - S_p(\lambda_0 f(u_0)) = u_0 - \lambda_0^{p^*} S_p(f(u_0)) = 0. \tag{4.14}$$

By Theorem 4.2 and the strict positivity of f, the function $F : (0, \infty) \times C^0[0, 1] \to C^0[0, 1]$ is C^2 (F is not even C^1 at $\lambda = 0$, if $p^* < 1$). We will denote the partial derivative of F with respect to λ by F_λ, and the partial Fréchet derivative of F with respect to u by F_u. Then, for $\bar{h}$, $\bar{h}_1$, $\bar{h}_2 \in C^0[0, 1]$,

$$\begin{aligned}
F_\lambda(\lambda_0, u_0) &= -p^*\lambda_0^{-1} S_p(\lambda_0 f(u_0)) = -p^*\lambda_0^{-1} u_0;\\
F_u(\lambda_0, u_0)\bar{h} &= \bar{h} - DS_p(\lambda_0 f(u_0))(\lambda_0 f'(u_0)\bar{h});\\
F_{uu}(\lambda_0, u_0)(\bar{h}_1, \bar{h}_2) &= -\big\{D^2 S_p(\lambda_0 f(u_0))(\lambda_0 f'(u_0)\bar{h}_1, \lambda_0 f'(u_0)\bar{h}_2)\\
&\qquad + DS_p(\lambda_0 f(u_0))(\lambda_0 f''(u_0)\bar{h}_1\bar{h}_2)\big\}.
\end{aligned} \tag{4.15}$$

We now characterise the null space and range of the operator $F_u(\lambda_0, u_0)$. We will denote the standard $L^2(0, 1)$ inner product by $\langle\cdot, \cdot\rangle$.

Proposition 4.5. *If $(\lambda_0, u_0) \in \widehat{\mathcal{S}}$ and $0 \neq w_0 \in N(F_u(\lambda_0, u_0))$ then w_0', $|u_0'|^{p-2}w_0' \in C^1[0, 1]$, and*

$$-(|u_0'|^{p-2}w_0')' = p^*\lambda_0 f'(u_0)w_0, \quad w_0'(0) = 0, \quad (|u'|^{p-2}w_0')|_0 = 0, \quad w_0(1) = 0, \tag{4.16}$$

$$R(F_u(\lambda_0, u_0)) \subset \{\psi \in C^0[0, 1] : \langle\psi, f'(u_0)w_0\rangle = 0\}. \tag{4.17}$$

In addition, w_0 has no zeros in $(0, 1)$.

Proof. The differentiability properties of w_0, and (4.16), follow from part (B) of Theorem 4.2 and (4.15). Now suppose that $\eta \in C^0[0, 1]$ and let $\psi = F_u(\lambda_0, u_0)\eta$. Then

$$\eta - \psi = DS_p(\lambda_0 f(u_0))(\lambda_0 f'(u_0)\eta),$$

so by part (B) of Theorem 4.2, $|u_0'|^{p-2}(\eta - \psi)' \in C^1[0, 1]$, and

$$-(|u_0'|^{p-2}(\eta - \psi)')' = p^*\lambda_0 f'(u_0)\eta, \quad (|u_0'|^{p-2}(\eta - \psi)')|_0 = 0, \quad \eta(1) - \psi(1) = 0. \tag{4.18}$$

Taking the inner product of the first equation in (4.18) with w_0 and integrating by parts (using the boundary conditions in (4.16) and (4.18)) yields

$$\begin{aligned}
\langle p^*\lambda_0 f'(u_0)w_0, \eta\rangle &= -\langle w_0, (|u_0'|^{p-2}(\eta - \psi)')'\rangle = -\langle(|u_0'|^{p-2}w_0')', \eta - \psi\rangle,\\
&= \langle p^*\lambda_0 f'(u_0)w_0, \eta - \psi\rangle \qquad \text{(by (4.16))}\\
&\implies \langle f'(u_0)w_0, \psi\rangle = 0,
\end{aligned}$$

which proves (4.17).

We note that, when $1 < p < 2$, the term $|u_0'|^{p-2}$ in the above calculations is singular at $x = 0$, but this term is always multiplied by a function that cancels out this singularity (recall Remark 4.3).

To prove the final part of the proposition we use a modification (and simplification) of the proof of [1, Lemma 3.6]. Define

$$\begin{aligned} D(L_{u_0}) &:= \{v \in C_0^1[0,1] : |u_0'|^{p-2}v' \in C^1[0,1]\}, \\ L_{u_0}v &:= (|u_0'|^{p-2}v')' + p^*\lambda_0 f'(u_0)v \in C^0(0,1], \quad v \in D(L_{u_0}). \end{aligned}$$

We note that the term $|u_0'|^{p-2}$ in the definition of $D(L_{u_0})$ may be singular at $x = 0$, so in this definition we mean that $v \in D(L_{u_0})$ if $|u_0'|^{p-2}v'$ coincides with an element of $C^1[0,1]$ on $(0,1]$; recall a similar remark in Remark 4.3 (b).

Lemma 4.6. *The functions* w_0, u_0, $(x-1)u_0' \in D(L_{u_0})$, *and*

$$L_{u_0}w_0 = 0, \tag{4.19}$$

$$L_{u_0}u_0 = -\lambda_0 f(u_0) + p^*\lambda_0 f'(u_0)u_0, \tag{4.20}$$

$$L_{u_0}((x-1)u_0') = -p^*p\lambda_0 f(u_0). \tag{4.21}$$

Proof. It follows immediately from Proposition 4.5 that $w_0 \in D(L_{u_0})$ and that (4.19) holds. Similarly, $u_0 \in D(\Delta_p)$ implies that $u_0 \in D(L_{u_0})$, and (4.20) follows immediately from (2.1). Next, differentiating (2.2) with respect to x on $(0,1)$ yields

$$\big\{|u_0'|^{p-2}u_0''\big\}' = -p^*\lambda_0 f'(u_0)u_0', \tag{4.22}$$

and hence,

$$\begin{aligned} \big\{|u_0'|^{p-2}((x-1)u_0')'\big\}' &= \big\{|u_0'|^{p-2}u_0' + (x-1)|u_0'|^{p-2}u_0''\big\}' \\ &= -(1+p^*)\lambda_0 f(u_0) + (x-1)\big\{|u_0'|^{p-2}u_0''\big\}' \qquad \text{(by (2.2))} \\ &= -p^*p\lambda_0 f(u_0) - p^*\lambda_0 f'(u_0)((x-1)u_0'). \qquad \text{(by (4.22))} \end{aligned}$$

Now, the right hand side lies in $C^0[0,1]$ so, by definition, $(x-1)u_0' \in D(L_{u_0})$, and (4.21) now follows from this result. This completes the proof of Lemma 4.6. □

Corollary 4.7. *Let*

$$\gamma := \max\Big\{\frac{\xi f'(\xi)}{f(\xi)} : 0 \leqslant \xi \leqslant |u_0|_0\Big\} > 0, \quad v_0 := u_0 + \gamma(x-1)u_0'.$$

Then $v_0 \in D(L_{u_0})$ *and*

$$L_{u_0}v_0 < 0 \ \ \textit{and} \ \ v_0(x) > 0 \ \ \textit{on} \ (0,1), \quad v_0(1) = 0. \tag{4.23}$$

Proof. By Lemma 4.6, $v_0 \in D(L_{u_0})$ and

$$\begin{aligned} L_{u_0}v_0 &= \lambda_0\big\{p^*f'(u_0)u_0 - (1+\gamma p^*p)f(u_0)\big\} \leqslant \lambda_0\big\{p^*\gamma - (1+\gamma p^*p)\big\}f(u_0) \\ &= -(1+\gamma)\lambda_0 f(u_0) < 0 \quad \text{on } (0,1). \end{aligned}$$

The other assertions regarding v_0 follow from Lemma 2.2. □

Now suppose that w_0 has at least one zero in $(0,1)$, and let x_0 be the maximum such zero. Then we may suppose that

$$w_0 > 0, \quad \text{on } (x_0,1), \quad w_0(x_0) = w_0(1) = 0, \quad w_0'(x_0) > 0, \quad w_0'(1) < 0. \tag{4.24}$$

We now consider the quantity

$$Q := \int_{x_0}^1 \big(w_0L_{u_0}v_0 - v_0L_{u_0}w_0\big).$$

The above results show immediately that $Q < 0$. However, by the definition of the domain $D(L_{u_0})$ and the fact that $x_0 > 0$, we can also use integration by parts to evaluate Q, and then (4.23)-(4.24) show that

$$Q = \big[|u_0'|^{p-2}\{w_0 v_0' - v_0 w_0'\}\big]_{x_0}^1 = |u_0'(x_0)|^{p-2} v_0(x_0) w_0'(x_0) > 0.$$

This contradiction shows that w_0 has no zero in $(0, 1)$, and so completes the proof of Proposition 4.5. □

As mentioned above, the remainder of the proof of Theorem 3.1 is relatively standard, so we only sketch it here. Further details are given in the proof of [1, Theorem 3.9].

We first consider the structure of the set of solutions of (4.14) near to $(\lambda_0, u_0) = (0, 0)$ in $\mathbb{R} \times C^1[0, 1]$. It is clear from the definition of F in (4.14) that there is a difficulty with the differentiability of F at $(0, 0)$, so instead we consider the problem

$$\tilde{F}(\mu, u) := u - \mu S_p(f(u)) = 0. \tag{4.25}$$

By Theorem 4.2 (and the properties of f), the function $\tilde{F} : \mathbb{R} \times C^1[0, 1] \to C^1[0, 1]$ is C^2, and $D_u\tilde{F}(0, 0) = Id$ (the identity on $C^1[0, 1]$). Hence, by the implicit function theorem, (4.25) has a C^2 solution of the form $\mu \to U(\mu)$, where $U : (-\epsilon, \epsilon) \to C^1[0, 0]$ is C^2, for some $\epsilon > 0$. That is, there is a unique curve of solutions in $\widehat{\mathcal{S}}$ emanating from $(0, 0)$, having the form described in part (B) of Theorem 3.1, which proves this part of the theorem.

Having 'started' the curve $\widehat{\mathcal{S}}$ from $(0, 0)$, we now extend it to infinity by a continuation argument. Consider an arbitrary point $(\lambda_0, u_0) \in \widehat{\mathcal{S}}$. Then

$$D_{(\lambda,u)}F(\lambda_0, u_0)(\bar{\lambda}, \bar{u}) = \bar{\lambda} F_\lambda(\lambda_0, u_0) + F_u(\lambda_0, u_0)\bar{u} = -\bar{\lambda} p^* \lambda_0^{-1} u_0 + F_u(\lambda_0, u_0)\bar{u}, \tag{4.26}$$

and by Theorem 4.2 and (4.15), $F_u(\lambda_0, u_0)$ is a compact perturbation of the identity on $C^0[0, 1]$. By Proposition 4.5, $\dim N(F_u(\lambda_0, u_0)) \leqslant 1$. If $\dim N(F_u(\lambda_0, u_0)) = 0$ then, by compactness, $F_u(\lambda_0, u_0)$ is nonsingular and so $D_{(\lambda,u)}F(\lambda_0, u_0)$ is surjective. If $\dim N(F_u(\lambda_0, u_0)) = 1$, then, by compactness, (4.17) holds with equality, and hence

$$\operatorname{codim} R(F_u(\lambda_0, u_0)) = 1, \quad u_0 \notin R(F_u(\lambda_0, u_0)), \tag{4.27}$$

so, by (4.26), $D_{(\lambda,u)}F(\lambda_0, u_0)$ is again surjective. Thus, in either case, the implicit function theorem shows that in a neighbourhood of (λ_0, u_0) the set $\widehat{\mathcal{S}}$ consists of a C^2 curve in $(0, \infty) \times C^0[0, 1]$ with a local parametrisation of the form $s : (-\epsilon, \epsilon) \to (\lambda(s), u(s))$, satisfying

$$(\lambda(0), u(0)) = (\lambda_0, u_0), \ \ F_u(\lambda(s), u(s))u_s(s) = \lambda_s(s) p^* \lambda(s)^{-1} u(s). \tag{4.28}$$

Hence, by (4.27) and (4.28),

$$\begin{aligned} F_u(\lambda_0, u_0) \text{ is singular } &\iff \lambda_s(0) = 0, \ u_s(0) \in N(F_u(\lambda_0, u_0)) \text{ has no zero in } (0, 1) \\ &\implies \lambda_{ss}(0) < 0 \end{aligned} \tag{4.29}$$

(given the above machinery and condition (1.4), the latter implication is a standard computation, which is described on [1, p. 2114]). Furthermore, it follows from (4.14) that $u(s) = S_p(\lambda(s) f(u(s)))$, $s \in (-\epsilon, \epsilon)$, so by Theorem 4.2, the local parametrisation $s : (-\epsilon, \epsilon) \to (\lambda(s), u(s))$ is C^2 into the space $(0, \infty) \times C^1[0, 1]$.

Standard arguments now show the following results.

- $\widehat{\mathcal{S}} \subset (0, \lambda_{\max}) \times P^0$, for some $\lambda_{\max} > 0$.
- Every component of $\widehat{\mathcal{S}}$ can be continued, as a smooth curve in $(0, \lambda_{\max}) \times P^0$, to $(0, 0)$ and 'to ∞', in the sense described in Theorem 3.1 (a). Since $\widehat{\mathcal{S}}$ consists of a single curve in a neighbourhood of $(0, 0)$, we conclude that $\widehat{\mathcal{S}}$ must have exactly one component, and it can be shown that this component can be parametrised by a single, global parametrisation of the form described in the theorem.
- It follows from Theorem 3.1 (a) that there exists at least one point $s^* \in (0, \infty)$ such that $\lambda_s(s^*) = 0$, and it follows from (4.29) that there is at most one such point s^*. This proves Theorem 3.1 (b).
- Part (c) now follows from parts (a) and (b), together with [1, Lemma 3.8].

This completes the proof of Theorem 3.1. □

4.3. **Proof of Corollary 3.2.** All the results of the corollary follow readily from Theorem 3.1, except (3.1). To prove this we first note that by Theorem 3.1 (c), Proposition 4.5 and (4.29), we have

$$u_s(s^*) > 0 \quad \text{on } (0,1), \tag{4.30}$$

so (3.1) holds on $(\lambda^* - \epsilon, \lambda^*)$, for some $\epsilon > 0$. Now suppose that (3.1) does not hold on $(0, \lambda^*)$. Then, by continuity, there exists $\lambda_0 \in (0, \lambda^*)$ and $x_0 \in [0,1]$ such that

$$u^-(\lambda_0)(x_0) = u^+(\lambda_0)(x_0), \quad u^-(\lambda_0)'(x_0) = u^+(\lambda_0)'(x_0).$$

But this implies that $u^-(\lambda_0) = u^+(\lambda_0)$, which contradicts Theorem 3.1 (c). □

5. Stability results

In this section we consider the following time-dependent, parabolic, initial-boundary value problem

$$\frac{dv}{dt} = \Delta_p(v) + \lambda f(v), \quad v(0) = v_0 \in C_0^0[-1,1], \tag{5.1}$$

where $\lambda > 0$. Clearly, the solutions $u^\pm(\lambda)$, $\lambda \in (0, \lambda^*)$, of (2.1) found in Corollary 3.2 can be regarded as equilibrium (constant in time) solutions of (5.1), and there are no other equilibria of (5.1). We will now discuss the local and global stability properties of these equilibrium solutions of (5.1).

We first define, briefly, what we mean by a solution of (5.1), and then state a standard result on the existence and uniqueness of such solutions. Further details are given in [9], but all these results are based on many preceding publications, see [4] for an overview of these. In this context we need to extend the domain of the operator Δ_p to an $L^2(-1,1)$ setting by defining

$$D(\Delta_p) := \{u \in C_0^1[-1,1] : \phi_p(u') \in W^{1,2}(-1,1)\},$$

so we now have $\Delta_p(u) = \phi_p(u')' \in L^2(-1,1)$, for $u \in D(\Delta_p)$.

Definition 5.1. For $0 < T \leqslant \infty$, let

$$\Sigma_T := C([0,T), C_0^0[-1,1]) \cap C((0,T), W_0^{1,p}(-1,1)) \cap W_{\text{loc}}^{1,2}((0,T), L^2(-1,1)).$$

A *solution* of (5.1) is a function $v \in \Sigma_T$, for some $T > 0$, such that $v(0) = v_0$ and for a.e. $t \in [0,T)$:

(a) $v(t) \in D(\Delta_p)$;
(b) the function $v : [0,T) \to L^2(-1,1)$ is differentiable at t;
(c) $\dfrac{dv}{dt}(t) = \Delta_p(v(t)) + \lambda f(v(t))$ (this equation holds in the $L^2(-1,1)$ sense).

Theorem 5.2. *For any $\lambda > 0$ and $v_0 \in C_0^0[-1,1]$, there exists a maximal time $T_{\lambda,v_0} > 0$ such that the problem* (5.1) *has a unique solution $v_{\lambda,v_0} \in \Sigma_{T_{\lambda,v_0}}$; the time T_{λ,v_0} is maximal, in the sense that*

$$T_{\lambda,v_0} < \infty \implies \lim_{t \nearrow T_{\lambda,v_0}} |v_{\lambda,v_0}(t)|_0 = \infty. \tag{5.2}$$

In addition, if $T_{\lambda,v_0} = \infty$ and there exists a sequence (t_n) in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\liminf_{n\to\infty} |v_{\lambda,v_0}(t_n)|_0 < \infty$, then, after choosing a subsequence if necessary, the limit $v_\infty = \lim_{n\to\infty} v_{\lambda,v_0}(t_n)$ exists in $W_0^{1,p}(-1,1)$, and v_∞ is an equilibrium of (5.1).

The final technical result in Theorem 5.2 is proved in the proof of [9, Theorem 4.1] (based on an argument in the proof of [4, Lemma 3.1]). It will be useful in proving our stability results below.

We now prove the local, linearised, exponential stability of the equilibrium $u^-(\lambda)$.

Theorem 5.3 (Local exponential stability)**.** *Suppose that $\lambda \in (0, \lambda^*)$. Then there exists κ, δ, $C > 0$ such that*

$$|v_0 - u^-(\lambda)|_0 < \delta \implies |v_{\lambda,v_0}(t) - u^-(\lambda)|_0 < Ce^{-\kappa t}, \quad t \geqslant 0.$$

Proof. We will use the linearised stability results of [10] so, for any $s > 0$ and $(\lambda(s), u(s)) \in \mathcal{S}$ as in Theorem 3.1, we consider the linearised eigenvalue problem

$$\begin{gathered}(p-1)(|u(s)'|^{p-2}w')' + \lambda(s)f'(u(s))w = \sigma w, \quad \text{on } (-1,1),\\ w(\pm 1) = 0.\end{gathered} \tag{5.3}$$

The differential operator in (5.3) is a linear, second order, formally self-adjoint Sturm-Liouville operator and, by (4.13), the coefficient function $|u(s)'|^{p-2}$ in (5.3) satisfies $1/|u(s)'|^{p-2} = |u(s)'|^{2-p} \in L^1(-1,1)$, which is the standard hypothesis in the L^1 theory of such operators, see [2, Chap. 8]. Thus, the eigenvalue problem (5.3) has all the standard properties described in [2]. We let $\sigma(s)$, $w(s)$ denote the principal eigenvalue and eigenfunction of (5.3), that is, $w(s)$ will denote the eigenfunction satisfying $w(s) > 0$ in $(-1,1)$, normalised by setting $|w(s)|_0 = 1$, and $\sigma(s)$ is the corresponding eigenvalue.

Now, $u(s)$ is an equilibrium of equation (5.1), with $\lambda = \lambda(s)$, and it follows from [10, Theorem 5.1] that:

(a) if $\sigma(s) < 0$ then the stability result of the theorem holds for $u(s)$;
(b) if $\sigma(s) > 0$ then $u(s)$ is unstable.

Hence, by the definition of the solutions $u^{\pm}(\lambda)$, it suffices to show that $\operatorname{sgn}\sigma(s) = \operatorname{sgn}(s - s^*)$ for all $s > 0$.

To do this we first note that, for each $s > 0$, the function $u(s) \in C_0^1[-1,1]$ is symmetric about 0. Hence, the principal eigenfunction of (5.3) is symmetric about 0 and coincides with the principal eigenvalue of the problem

$$\begin{gathered}(p-1)(|u(s)'|^{p-2}w')' + \lambda(s)f'(u(s))w = \sigma w, \quad \text{on } (0,1),\\ (|u(s)'|^{p-2}w')|_0 = 0, \quad w(1) = 0,\end{gathered} \tag{5.4}$$

where the boundary condition at $x = 0$ in (5.4) is the standard Neumann-type condition in the L^1 theory of this type of problem, see [2]. Considering the eigenvalue problem (5.4) rather than (5.3) will enable us to utilise the results of Section 3.

Differentiating the equation $u(s) = S_p(\lambda(s)f(u(s))$ with respect to s, and using (4.10), yields

$$\begin{gathered}-(p-1)(|u(s)'|^{p-2}u_s(s)')' = \lambda_s(s)f(u(s)) + \lambda(s)f'(u(s))u_s(s),\\ |u(s)'|^{p-2}u_s(s)'|_0 = 0, \quad u_s(s)|_1 = 0.\end{gathered} \tag{5.5}$$

Now, taking the inner product of (5.3) and (5.5) with $u_s(s)$ and $w(s)$, respectively, and integrating by parts yields

$$\sigma(s)\langle u_s(s), w(s)\rangle = -\lambda_s(s)\langle f(u(s)), w(s)\rangle. \tag{5.6}$$

Hence, by the positivity of f and w, and the properties of $\lambda(s)$ in Theorem 3.1, $\sigma(s^*) = 0$, and s^* is the only zero of $\sigma(\cdot)$. Also, by (4.30) and the continuity of $u_s(\cdot)$ in $C^1[0,1]$, there exists $\epsilon > 0$ such that, for $|s - s^*| < \epsilon$, $u_s(s) > 0$ on $(0,1)$, so by (5.6),

$$\operatorname{sgn}\sigma(s) = -\operatorname{sgn}\lambda_s(s) = \operatorname{sgn}(s - s^*), \text{ for } 0 < |s - s^*| < \epsilon, \tag{5.7}$$

and since $s = s^*$ is the only zero of both the functions $\sigma(\cdot)$ and $\lambda(\cdot)$ on $(0,\infty)$, (5.7) holds for all $s > 0$. This completes the proof of Theorem 5.3. □

Remark 5.4. The proof of Theorem 5.3, together with [10, Theorem 5.1], also shows that, locally, $u^+(\lambda)$ is exponentially unstable, in the sense described in [10]. This result is rather more complicated to describe than the stability result in Theorem 5.3, and less interesting, so we have omitted the details here – they can be found in [10]. The instability of $u^+(\lambda)$ will also be described in Theorem 5.5.

We now describe more global behaviour of the solutions of (5.1). We first note that, by the comparison theorem [10, Theorem 4.4], for any $\lambda > 0$ and $v_0 \in C_0^0[-1,1]$,

$$v_{\lambda,v_0}(t) \geqslant \min\{v_0(x) : x \in [-1,1]\}, \quad t \geqslant 0,$$

that is, v_{λ,v_0} is bounded below. Also, we recall from Corollary 3.2 that $u^-(\lambda) < u^+(\lambda)$, on $(-1,1)$, for $0 < \lambda < \lambda^*$.

Theorem 5.5. (*a*) *Suppose that* $0 < \lambda < \lambda^*$. *Then, for any* $\delta > 0$ *there exists* $\underline{S}_{\lambda,\delta}$, $\overline{S}_{\lambda,\delta} \in C_0^1[-1,1]$ *such that*:

(*i*) $\underline{S}_{\lambda,\delta} < u^+(\lambda) < \overline{S}_{\lambda,\delta}$ *on* $(-1,1)$, *and* $|\overline{S}_{\lambda,\delta} - \underline{S}_{\lambda,\delta}|_1 \leqslant \delta$;

(*ii*) *if* $v_0 \leqslant \underline{S}_{\lambda,\delta}$ *then:*

$$v_{\lambda,v_0}(t) \leqslant \underline{S}_{\lambda,\delta} \text{ for all } t \geqslant 0, \quad T_{\lambda,v_0} = \infty \quad \text{and} \quad \lim_{t\to\infty} \|v_{\lambda,v_0}(t) - u^-(\lambda)\|_{1,p} = 0;$$

(*iii*) *if* $v_0 \geqslant \overline{S}_{\lambda,\delta}$ *then:*

$$v_{\lambda,v_0}(t) \geqslant \overline{S}_{\lambda,\delta} \text{ for all } t \geqslant 0 \quad \text{and} \quad \lim_{t \nearrow T_{\lambda,v_0}} |v_{\lambda,v_0}(t)|_0 = \infty.$$

(*b*) *Suppose that* $\lambda > \lambda^*$. *Then, for any* v_0, $\lim_{t \nearrow T_{\lambda,v_0}} |v_{\lambda,v_0}(t)|_0 = \infty$.

Proof. (a) Suppose that $0 < \lambda < \lambda^*$. By combining the results obtained in the proof of Theorem 5.3 above with the proof of [10, Theorem 5.1], we can construct comparison functions $\underline{S}_{\lambda,\delta}$, $\overline{S}_{\lambda,\delta}$ having the properties (in particular, the inequalities) described in the theorem (the notation $S_\delta^\pm$ is used in [10], but in the current setting this would be slightly confusing with the notation $u^\pm(\lambda)$ for the equilibria). Specifically, to construct $\overline{S}_{\lambda,\delta}$ we follow the construction of S_δ^+ in the proof of part (b) of [10, Theorem 5.1], setting the parameter κ there to be zero; $\underline{S}_{\lambda,\delta}$ can be constructed in the same manner.

Now suppose that $v_0 \leqslant \underline{S}_{\lambda,\delta}$. Then

$$-|v_0|_0 \leqslant v_{\lambda,v_0}(\cdot) \leqslant \underline{S}_{\lambda,\delta} < u^+(\lambda), \tag{5.8}$$

so $|v_{\lambda,v_0}(\cdot)|_0$ is bounded and, by (5.2), $T_{\lambda,v_0} = \infty$. Now suppose that there exists a sequence (t_n) in $(0,\infty)$ and $\epsilon > 0$ such that $t_n \to \infty$ and $\|v_{\lambda,v_0}(t_n) - u^-(\lambda)\|_{1,p} > \epsilon$, $n = 1, 2, \dots$. Then, by the final result in Theorem 5.2, after choosing a subsequence if necessary, $v_{\lambda,v_0}(t_n) \to v_\infty$ in $W_0^{1,p}(-1,1)$, where v_∞ is an equilibrium of (5.1). But the only equilibria of (5.1) are $u^\pm(\lambda)$, and the construction of v_∞ precludes $v_\infty = u^-(\lambda)$, while (5.8) precludes $v_\infty = u^+(\lambda)$. This contradiction completes the proof of part (a)-(ii).

Next, suppose that $v_0 \geqslant \overline{S}_{\lambda,\delta}$. If $T_{\lambda,v_0} < \infty$ then the result follows from (5.2), so let us also suppose that $T_{\lambda,v_0} = \infty$ and $\liminf_{t\to\infty} |v_{\lambda,v_0}(t)|_0 < \infty$. Then, by Theorem 5.2, there exists a sequence (t_n) such that $t_n \to \infty$ and $v_{\lambda,v_0}(t_n)$ converges to an equilibrium. But this is impossible, since $u^\pm(\lambda) < \overline{S}_{\lambda,\delta} \leqslant v_{\lambda,v_0}(\cdot)$, which proves part (a)-(iii).

(b) The proof is similar to the proof of part (a)-(iii) (in this case there are no equilibria for v_{λ,v_0} to converge to), which completes the proof of Theorem 5.5. □

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